

Stochastic sensitivity analysis: theory and numerical algorithms

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Abstract. A problem of the analysis of nonlinear dynamic systems forced by the random disturbances is considered. For the description of probabilistic distributions around the deterministic attractors, a theory of the quasipotential and its approximations based on the stochastic sensitivity functions (SSF) technique is suggested. Constructive methods for the computation of SSF for the equilibria and limit cycles are presented and discussed. We show how SSF can be used for the visual description of probabilistic distributions in the form of confidence domains.

1. Introduction

Currently, the stochastic dynamic systems are widely used for the modeling of processes in various fields of science [1]. In nonlinear systems, inevitably presented random disturbances can cause some unexpected and complex probabilistic phenomena [2, 3]. A direct numerical simulation is still a main tool of the investigation of these phenomena. However, in parametric studies, this method is extremely time-consuming. So, the analytical methods of the probabilistic analysis attract attention of many researchers.

In this paper, for the analysis of nonlinear stochastic systems, we set forth a new constructive approach based on the stochastic sensitivity functions and confidence domains.

Consider a general stochastic system of nonlinear differential equations

$$dx = f(x)dt + \varepsilon\sigma(x)dw, \quad (1)$$

where x is an n -dimensional vector, $f(x)$ is an n -vector function, $\sigma(x)$ is an $n \times m$ -matrix function, $w(t)$ is an m -dimensional Wiener process, and ε is a scalar parameter of the noise intensity.

It is supposed that the corresponding deterministic system (1) with $\varepsilon = 0$ has an exponentially stable attractor A . It means that for the small neighbourhood D of the attractor A , there exist constants $K > 0$, $l > 0$ such that for any solution $x(t)$ of the deterministic system with $x(0) = x_0 \in D$ the following inequality holds

$$\|\Delta(x(t))\| \leq Ke^{-lt}\|\Delta(x_0)\|.$$

Here, $\Delta(x) = x - \gamma(x)$ is a vector of a deviation of the point x from the attractor A , $\gamma(x)$ is a point of the attractor A that is nearest to x .



As a result of stochastic forcing, the random trajectories of system (1) form some flow with the probability density function $\rho(t, x, \varepsilon)$. This function is governed by the Kolmogorov-Fokker-Planck equation [4]

$$\frac{\partial \rho}{\partial t} = L\rho, \quad L\rho = \frac{\varepsilon^2}{2} \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} (a_{ij}\rho) - \sum_{i=1}^n \frac{\partial}{\partial x_i} (f_i\rho), \quad a_{ij} = [\sigma\sigma^\top]_{ij}.$$

If a character of the transient is unessential and the main interest is focused on the stable stationary regime, then it is possible to restrict a stochastic analysis by the study of the stationary density function $\rho(x, \varepsilon)$. This function is a solution of the stationary Kolmogorov-Fokker-Planck equation

$$L\rho = 0.$$

The analysis of this equation is a very difficult problem technically even for the 2-dimensional case. So, the asymptotics based on the quasipotential

$$v(x) = -\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \ln \rho(x, \varepsilon)$$

are commonly used [5].

The quasipotential is connected with some variational problem of the minimization of the action potential and governed by the Hamilton-Jacobi equation

$$\left(f(x), \frac{\partial v}{\partial x} \right) + \frac{1}{2} \left(\frac{\partial v}{\partial x}, \sigma(x)\sigma^\top(x) \frac{\partial v}{\partial x} \right) = 0$$

with conditions $v|_A = 0$, $v|_{D \setminus A} > 0$.

The solution of this equation is still difficult problem, therefore in a small neighbourhood of the attractor A , a first quadratic approximation of the function $v(x)$ is used [6]

$$v(x) \approx \frac{1}{2} (\Delta(x), \Phi(\gamma(x))\Delta(x)), \quad \Phi(x) = \frac{\partial^2 v}{\partial x^2}(x).$$

In what follows, we will present constructive methods of such approximations for equilibria and cycles.

2. Stochastic sensitivity of equilibria

Let the attractor of the deterministic system be an equilibrium \bar{x} . In this case, the quadratic approximation $v(x) \approx \frac{1}{2}(x - \bar{x}, \Phi(\bar{x})(x - \bar{x}))$ of the quasipotential gives the following Gaussian approximation of $\rho(x, \varepsilon)$ in a neighbourhood of the equilibrium \bar{x}

$$\rho(x, \varepsilon) \approx N \exp \left(-\frac{(x - \bar{x}, W^{-1}(x - \bar{x}))}{2\varepsilon^2} \right), \quad W = \Phi^{-1}(\bar{x})$$

with the mean \bar{x} , the covariance matrix $C = \varepsilon^2 W$, and the normalization constant, N . Here, the matrix W is a solution of the equation

$$FW + WF^\top + S = 0, \tag{2}$$

where $F = \frac{\partial f}{\partial x}(\bar{x})$ is the Jacobi matrix of the deterministic system at \bar{x} and $S = \sigma(\bar{x})\sigma^\top(\bar{x})$. For the exponentially stable equilibrium \bar{x} , the eigenvalues of the Jacobi matrix F have the negative

real parts, and the matrix equation 2 has a unique solution being the stochastic sensitivity matrix of the equilibrium [7].

For the low-dimensional cases, it is convenient to find the solution of system (2) by direct numerical methods. In the case of systems with large dimensions, the iterative methods are more suitable. As an example, consider the iterative method associated with the procedure of the stabilization of solutions for the corresponding system of differential equations.

The matrix W (required stochastic sensitivity function of the equilibrium \bar{x}) as a solution of equation (2) can be obtained by the following limit procedure.

Let matrix function $V(t)$ be an arbitrary solution of the differential equation

$$\dot{V} = FV + VF^\top + S$$

on the interval $[0, +\infty)$. If the equilibrium is exponentially stable then the matrix $V(t)$ tends to the matrix W as $t \rightarrow +\infty$

$$\lim_{t \rightarrow +\infty} V(t) = W.$$

Note that the convergence rate depends on degree of the equilibrium stability.

To visualize a spatial probabilistic distribution of random states around the equilibrium \bar{x} , one can construct a confidence ellipsoid:

$$(x - \bar{x}, W^{-1}(x - \bar{x})) = \varepsilon^2 K(P),$$

where P is a fiducial probability. The function $K(P)$ is an inverse function to $P(K)$:

$$P(K) = \frac{\Phi_n(K)}{\Phi_n(\infty)}, \quad \Phi_n(K) = \int_0^{\sqrt{K}} e^{-\frac{t^2}{2}} t^{n-1} dt.$$

In the one-dimensional case ($n = 1$),

$$P(K) = \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{K}} e^{-\frac{t^2}{2}} dt = \operatorname{erf}\left(\sqrt{\frac{K}{2}}\right), \quad \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt,$$

and the corresponding confidence interval $(\bar{x} - r, \bar{x} + r)$ is defined by $r = \varepsilon \sqrt{2\mu} \operatorname{erf}^{-1}(P)$, where the stochastic sensitivity μ can be found from the explicit formula:

$$\mu = -\frac{\sigma^2(\bar{x})}{2f'(\bar{x})}.$$

For the 3σ -rule, it holds that $r = 3\varepsilon\sqrt{\mu}$.

In the two-dimensional case ($n = 2$),

$$P(K) = 1 - e^{-\frac{K}{2}}, \quad K(P) = -2 \ln(1 - P).$$

The equation of the confidence ellipse can be written as

$$\frac{\beta_1^2}{\mu_1} + \frac{\beta_2^2}{\mu_2} = \varepsilon^2 K(P), \quad \beta_1 = (x - \bar{x}, v_1), \quad \beta_2 = (x - \bar{x}, v_2). \quad (3)$$

Here, μ_1, μ_2 are the eigenvalues, and v_1, v_2 are the normalized eigenvectors of the stochastic sensitivity matrix W . These vectors define the directions of the confidence ellipse axis, and μ_1, μ_2 define the values of corresponding semi-axis.

In the three-dimensional case ($n = 3$),

$$P(K) = \sqrt{\frac{2}{\pi}} \left[\int_0^{\sqrt{K}} e^{-\frac{t^2}{2}} dt - \sqrt{K} e^{-\frac{K}{2}} \right] = \operatorname{erf} \left(\sqrt{\frac{K}{2}} \right) - \sqrt{\frac{2K}{\pi}} e^{-\frac{K}{2}}.$$

The confidence ellipsoid can be written in the following form

$$\frac{\beta_1^2}{\mu_1} + \frac{\beta_2^2}{\mu_2} + \frac{\beta_3^2}{\mu_3} = \varepsilon^2 K(P), \quad (4)$$

where $\beta_1 = (x - \bar{x}, v_1)$, $\beta_2 = (x - \bar{x}, v_2)$, $\beta_3 = (x - \bar{x}, v_3)$. Here, μ_1, μ_2, μ_3 are the eigenvalues, and v_1, v_2, v_3 are the normalized eigenvectors of the stochastic sensitivity matrix W .

For $n = 4$, we have $P(K) = 1 - e^{-\frac{K}{2}} (1 + 0.5K)$.

3. Stochastic sensitivity of limit cycles

Now, let the attractor of the deterministic system be an exponentially stable limit cycle defined by a T -periodic solution $\bar{x}(t)$, $\bar{x}(t+T) = \bar{x}(t)$. Let Π_t be a hyperplane orthogonal to the cycle at the point $\bar{x}(t)$. For the Poincare section Π_t in the neighbourhood of the point $\bar{x}(t)$, the quadratic approximation

$$v(x)|_{\Pi_t} \approx \frac{1}{2} (x - \bar{x}(t), \Phi(\bar{x}(t))(x - \bar{x}(t)))$$

of the quasipotential gives the approximation $\rho_t(x, \varepsilon)$ of the probability density $\rho(x, \varepsilon)|_{\Pi_t}$ in the Gaussian form

$$\rho_t(x, \varepsilon) = N \exp \left(-\frac{(x - \bar{x}(t), W^+(t)(x - \bar{x}(t)))}{2\varepsilon^2} \right), \quad W(t) = \Phi^+(\bar{x}(t))$$

with the mean $\bar{x}(t)$ and covariance matrix $C(t) = \varepsilon^2 W(t)$. Here, the sign "+" means the pseudoinversion.

For the exponentially stable limit cycle, the matrix function $W(t)$ is a unique solution of the boundary problem [8]

$$\dot{W} = F(t)W + WF^\top(t) + P(t)S(t)P(t), \quad (5)$$

$$W(0) = W(T), \quad (6)$$

$$W(t)r(t) \equiv 0. \quad (7)$$

Here,

$$F(t) = \frac{\partial f}{\partial x}(\bar{x}(t)), \quad S(t) = \sigma(\bar{x}(t))\sigma^\top(\bar{x}(t)), \quad r(t) = f(\bar{x}(t)), \quad P(t) = P_{r(t)},$$

and $P_r = I - rr^\top / r^\top r$ is a projection matrix onto the hyperplane that is orthogonal to the vector r .

3.1. How to calculate the stochastic sensitivity of two-dimensional cycle

For the 2D-case, the stochastic sensitivity matrix $W(t)$ can be written as $W(t) = \mu(t)P(t)$. The scalar function $\mu(t) > 0$ is a T -periodic stochastic sensitivity function satisfying the following boundary problem [7]

$$\dot{\mu} = a(t)\mu + b(t), \quad \mu(0) = \mu(T) \quad (8)$$

with the T -periodic coefficients

$$a(t) = p^\top(t)(F^\top(t) + F(t))p(t), \quad b(t) = p^\top(t)S(t)p(t), \quad P(t) = p(t)p^\top(t),$$

where $p(t)$ is a normalized vector orthogonal to $f(\bar{x}(t))$. An explicit formula for the solution $\mu(t)$ of problem (8) is given by

$$\mu(t) = g(t)(c + h(t)),$$

where

$$g(t) = \exp\left(\int_0^t a(s)ds\right), \quad h(t) = \int_0^t \frac{b(s)}{g(s)}ds, \quad c = \frac{g(T)h(T)}{1 - g(T)}.$$

The value $M = \max \mu(t)$, $t \in [0, T]$ plays an important role in the analysis of stochastic dynamics near the limit cycle. We shall consider M as a *stochastic sensitivity factor* of the cycle.

Note that the function $\mu(t)$ allows us to construct a confidence band around the deterministic cycle. For the line Π_t that is orthogonal to the cycle at the point $\bar{x}(t)$, the boundaries $x_{1,2}(t)$ of the confidence band can be written in an explicit parametric form:

$$x_{1,2}(t) = \bar{x}(t) \pm \varepsilon \sqrt{2\mu(t)} \operatorname{erf}^{-1}(P)p(t). \quad (9)$$

The confidence bands are sufficiently simple and evident models for the spatial description of random states around the deterministic cycle.

3.2. How to calculate the stochastic sensitivity of three-dimensional cycle

In the three-dimensional case ($n = 3$), a singular decomposition of the symmetric non-negative defined 3×3 -matrix $V(t)$ looks like

$$V(t) = \lambda_1(t)v_1(t)v_1^\top(t) + \lambda_2(t)v_2(t)v_2^\top(t) + \lambda_3(t)v_3(t)v_3^\top(t),$$

where $\lambda_1(t) \geq \lambda_2(t) \geq \lambda_3(t)$ are the eigenvalues, and $v_1(t), v_2(t), v_3(t)$ are the normalized eigenvectors of matrix $V(t)$.

From condition (7) it follows that for any t , the matrix $V(t)$ is degenerated (a distribution of intersection points is concentrated in a hyperplane Π_t). It means that $\lambda_3(t) \equiv 0$, and the appropriate eigenvector $v_3(t) = r(t)/\|r(t)\|$ is tangent to the cycle. It allows us to write the decomposition of the matrix $V(t)$ as follows

$$V(t) = \lambda_1(t)v_1(t)v_1^\top(t) + \lambda_2(t)v_2(t)v_2^\top(t). \quad (10)$$

Here $V(t)$ is determined by the scalar functions $\lambda_1(t), \lambda_2(t)$ and vectors $v_1(t), v_2(t)$. In the case of non-degenerate noises, the functions $\lambda_1(t), \lambda_2(t)$ are strictly positive and determine for any t a dispersion of random trajectories around cycle along vectors $v_1(t), v_2(t)$. The values $\lambda_1(t), \lambda_2(t)$ determine the size and $v_1(t), v_2(t)$ determine the directions of the dispersion ellipse axes. The equation of this ellipse in plane Π_t looks like

$$(x - \bar{x}(t))^\top W^+(t)(x - \bar{x}(t)) = -2\varepsilon^2 \ln(1 - P),$$

where P is a fiducial probability. A set of these confidence ellipses along the deterministic limit 3D-cycle forms a confidence torus (see details in [9]).

Denote by $u_1(t), u_2(t)$ some orthonormal basis of the plane Π_t . One can easily find this basis if the T -periodic solution $\bar{x}(t)$ is known (see Remark). The eigenvectors $v_1(t), v_2(t)$, can be represented by rotation of basis $u_1(t), u_2(t)$ with some angle $\varphi(t)$

$$\begin{aligned} v_1(t) &= u_1(t) \cos \varphi(t) + u_2(t) \sin \varphi(t), \\ v_2(t) &= -u_1(t) \sin \varphi(t) + u_2(t) \cos \varphi(t). \end{aligned} \quad (11)$$

Thus, the decomposition (10), (11) allows us to express an unknown solution of the system (5)-(7) by means of three scalar functions $\lambda_1(t)$, $\lambda_2(t)$, $\varphi(t)$.

Denote

$$P_1(t) = v_1(t) \cdot v_1^\top(t), \quad P_2(t) = v_2(t) \cdot v_2^\top(t).$$

Remark that $P_i(t)$ ($i = 1, 2$) are the projective matrices

$$P_i v_i = v_i, \quad P_i v_j = 0 \quad (i \neq j). \quad (12)$$

Rewrite the decomposition (10) as follows

$$V(t) = \lambda_1(t) \cdot P_1(t) + \lambda_2(t) \cdot P_2(t).$$

Theorem 1. [8] *The matrix $V(t)$ is a solution of system (5),(6) if and only if the scalar functions $\lambda_1(t)$, $\lambda_2(t)$, $\varphi(t)$ of decomposition (10), (11) satisfy the following system*

$$\begin{aligned} \dot{\lambda}_1 &= \lambda_1 v_1^\top [F + F^\top] v_1 + v_1^\top S v_1 \\ \dot{\lambda}_2 &= \lambda_2 v_2^\top [F + F^\top] v_2 + v_2^\top S v_2 \\ (\lambda_1 - \lambda_2) \dot{\varphi} &= \lambda_2 v_1^\top F v_2 + \lambda_1 v_1^\top F^\top v_2 + v_1^\top S v_2 - (\lambda_1 - \lambda_2) \dot{u}_1^\top u_2. \end{aligned} \quad (13)$$

As we see, a construction of the solution $V(t)$ of system (5),(6) on the basis of a singular decomposition is reduced to the solution of system (13) for three scalar functions. The matrix $W(t)$ (required stochastic sensitivity function of a cycle) of the solution of system (5)-(7) can be obtained by the following limit procedure.

Theorem 2. [8] *Let the T -periodic matrix $W(t)$ be a solution of system (5)-(7). Let $\lambda_1(t)$, $\lambda_2(t)$, $\varphi(t)$ be an arbitrary solution of system (13) on the interval $[0, +\infty)$. Put $V(t) = \lambda_1(t) \cdot P_1(t) + \lambda_2(t) \cdot P_2(t)$, where $P_i(t) = v_i(t) v_i^\top(t)$ with the vector functions $v_i(t)$ obtained from (11). Then the matrix $V(t)$ tends to the matrix $W(t)$ as $t \rightarrow +\infty$*

$$\lim_{t \rightarrow +\infty} (V(t) - W(t)) = 0.$$

Remark. For the computation of system (13) right parts, it is necessary to know the vector functions $u_1(t)$, $u_2(t)$ and $\dot{u}_1(t)$. To find these, the following method is suggested.

Rewrite the initial 3D-system $\dot{x} = f(x)$ and its T -periodic solution $x = \bar{x}(t)$ by coordinates

$$\dot{x}_1 = f_1(x_1, x_2, x_3), \quad \dot{x}_2 = f_2(x_1, x_2, x_3), \quad \dot{x}_3 = f_3(x_1, x_2, x_3),$$

$$\bar{x}(t) = (\bar{x}_1(t), \bar{x}_2(t), \bar{x}_3(t))^\top.$$

A tangent vector $r(t) = (f_1(t), f_2(t), f_3(t))^\top$ has the coordinates $f_i(t) = f_i(\bar{x}(t))$, $i = 1, 2, 3$. One can choose the vectors $u_1(t), u_2(t)$ of the orthonormal basis for the plane Π_t in the following form

$$u_1 = g_1 \cdot \begin{pmatrix} -f_2 \\ f_1 \\ 0 \end{pmatrix}, \quad u_2 = g_2 \cdot \begin{pmatrix} -f_1 f_3 \\ -f_2 f_3 \\ f_1^2 + f_2^2 \end{pmatrix},$$

where

$$g_1 = (f_1^2 + f_2^2)^{-\frac{1}{2}}, \quad g_2 = \left(f_3^2 (f_1^2 + f_2^2) + (f_1^2 + f_2^2)^2 \right)^{-\frac{1}{2}}.$$

As a result of such choice the following formula holds

$$\dot{u}_1 = \dot{g}_1 \cdot \begin{pmatrix} -f_2 \\ f_1 \\ 0 \end{pmatrix} + g_1 \cdot \begin{pmatrix} -\frac{\partial f_2}{\partial x_1} & -\frac{\partial f_2}{\partial x_2} & -\frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix},$$

where

$$\dot{g}_1 = -\left(f_1^2 + f_2^2\right)^{-\frac{3}{2}} \cdot \left(f_1 \cdot \left(\frac{\partial f_1}{\partial x_1} f_1 + \frac{\partial f_1}{\partial x_2} f_2 + \frac{\partial f_1}{\partial x_3} f_3\right) + f_2 \cdot \left(\frac{\partial f_2}{\partial x_1} f_1 + \frac{\partial f_2}{\partial x_2} f_2 + \frac{\partial f_2}{\partial x_3} f_3\right)\right).$$

4. A development of the theory. Applications

In continuous-time dynamic systems, along with equilibria and limit cycles, the invariant tori are an important type of attractors. An extension of SSF technique for the case of tori is given in [10].

For the discrete-time systems, SSF technique was elaborated for such regular attractors as equilibria, cycles [11] and closed invariant curves [12]. For the description of probabilistic distributions near chaotic attractors, SSF approach is presented in [13].

Some constructive abilities of the SSF technique and confidence domains method were demonstrated in applications to the wide range of noise-induced phenomena: noise-induced transitions between attractors [7, 14]; stochastic deformations of oscillatory regimes in period-doubling bifurcations [15, 16]; backward stochastic bifurcations [17]; noise-induced chaos-order transitions [18].

Our approach clarified probabilistic mechanisms of these stochastic phenomena in the population systems [7, 19], neuron activity [20, 21, 22], and climate dynamics [14, 23].

The SSF technique was successfully used to solve the engineering problems of stabilization of stochastically forced operating modes and controlling of chaos [24, 25].

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